THE BUSEMANN-PETTY PROBLEM FOR ARBITRARY MEASURES.

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ABSTRACT. The aim of this paper is to study properties of sections of convex bodies with respect to different types of measures. We present a formula connecting the Minkowski functional of a convex symmetric body K with the measure of its sections. We apply this formula to study properties of general measures most of which were known before only in the case of the standard Lebesgue measure. We solve an analog of the Busemann-Petty problem for the case of general measures. In addition, we show that there are measures, for which the answer to the generalized Busemann-Petty problem is affirmative in all dimensions. Finally, we apply the latter fact to prove a number of different inequalities concerning the volume of sections of convex symmetric bodies in $\mathbb{R}^n$ and solve a version of generalized Busemann-Petty problem for sections by k-dimensional subspaces.

1. INTRODUCTION

Consider a non-negative, even function $f_n(x)$, which is locally integrable on $\mathbb{R}^n$. Let μ_n be the measure on $\mathbb{R}^n$ with density f_n.

For $\xi \in S^{n-1}$, let $\xi^\perp$ be the central hyperplane orthogonal to ξ. Define a measure μ_{n-1} on $\xi^\perp$, for each $\xi \in S^{n-1}$, so that for every bounded Borel set $B \subset \xi^\perp$,

$$\mu_{n-1}(B) = \int\limits_B f_{n-1}(x)dx,$$

where f_{n-1} is an even function on $\mathbb{R}^n$, which is locally integrable on each $\xi^\perp$.

In this paper we study the following problem

The Busemann-Petty problem for general measures (BPGM): *Fix $n \geq 2$. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that*

$$\mu_{n-1}(K \cap \xi^\perp) \leq \mu_{n-1}(L \cap \xi^\perp)$$

for every $\xi \in S^{n-1}$, does it follow that

$$\mu_n(K) \leq \mu_n(L)?$$

Clearly, the BPGM problem is a triviality for $n = 2$ and $f_{n-1} > 0$, and the answer is "yes", moreover $K \subseteq L$. Also note that this problem is a generalization of the Busemann-Petty problem, posed in 1956 (see [BP])

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and asking the same question for Lebesgue measures: $\mu_n(K) = \mathrm{Vol}_n(K)$ and $\mu_{n-1}(K \cap \xi^\perp) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp)$ i.e. $f_n(x) = f_{n-1}(x) = 1$.

Minkowski's theorem (see [Ga3]) shows that an origin-symmetric star-shaped body is uniquely determined by the volume of its hyperplane sections (the same is true for the case of general symmetric measure, see Corollary 1 below). In view of this fact it is quite surprising that the answer to the original Busemann Petty problem is negative for $n \geq 5$. Indeed, it is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of a sequence of papers: [LR] $n \geq 12$, [Ba] $n \geq 10$, [Gi] and [Bo2] $n \geq 7$, [Pa] and [Ga1] $n \geq 5$, [Ga2] $n = 3$, [Zh2] and [GKS] $n = 4$ (we refer to [Zh2], [GKS] and [K8] for more historical details).

It was shown in [Z], that the answer to BPGM in the case of the standard Gaussian measure ($f_n(x) = f_{n-1} = e^{-|x|^2/2}$) is the same: affirmative if $n \leq 4$ and negative if $n \geq 5$.

Answers to the original and Gaussian Busemann-Petty problems suggest that the answer to the BPGM could be independent from the choice of measures and depend only on the dimension n.

In Corollary 2 below we confirm this conjecture by proving the following:

Let $f_n(x) = f_{n-1}(x)$ be equal even nonnegative continuous functions, then the answer to the BPGM problem is affirmative if $n \leq 4$ and negative if $n \geq 5$.

Actually, the above fact is a corollary of a pair of more general theorems. Those theorems use the Fourier transform in the sense of distributions to characterize those functions $f_n(x)$ and $f_{n-1}(x)$ for which the BPGM problem has affirmative (or negative) answer in a given dimension:

Theorem 1. (BPGM: affirmative case) *Let f_n and f_{n-1} be even continuous nonnegative functions such that*

$$t\frac{f_n(tx)}{f_{n-1}(tx)} \tag{1}$$

is an increasing function of t for any fixed $x \in S^{n-1}$. Consider a symmetric star-shaped body K in $\mathbb{R}^n$ such that

$$\|x\|_K^{-1}\frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} \tag{2}$$

is a positive definite distribution on $\mathbb{R}^n$. Then for any symmetric star-shaped body L in $\mathbb{R}^n$ satisfying

$$\mu_{n-1}(K \cap \xi^\perp) \leq \mu_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

we have

$$\mu_n(K) \leq \mu_n(L).$$

Theorem 2. (BPGM: negative case) *Let f_n and f_{n-1} be even continuous nonnegative functions such that*

$$t\frac{f_n(tx)}{f_{n-1}(tx)}$$

is an increasing function of t for any fixed $x \in S^{n-1}$. Also assume that $f_{n-1}(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and is it strictly positive on $\mathbb{R}^n \setminus \{0\}$. If L is an infinitely smooth, origin symmetric, convex body in $\mathbb{R}^n$ with positive curvature, and the function

$$\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} \tag{3}$$

is in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and does not represent a positive definite distribution, then there exists a convex symmetric body D in $\mathbb{R}^n$ such that

$$\mu_{n-1}(D \cap \xi^\perp) \le \mu_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

but

$$\mu_n(D) > \mu_n(L).$$

Note that the differentiability assumptions in Theorem 2 and the assumption on f_{n-1} to be strictly positive are not critical for most of applications and can be avoided by applying a standard approximation argument (see Section 4).

Theorems 1 and 2 are generalizations of a theorem of Lutwak (see [Lu]) who provided a characterization of symmetric star-shaped bodies for which the original Busemann-Petty problem has an affirmative answer (see [Z] for the case of Gaussian measure). Let K and M be symmetric star-shaped bodies in $\mathbb{R}^n$. We say that K is the intersection body of M if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction. A more general class of *intersection bodies* is defined as the closure in the radial metric of the class of intersection bodies of star-shaped bodies (see [Ga3], Chapter 8).

Lutwak ([Lu], see also [Ga2] and [Zh1]) proved that if K is an intersection body then the answer to the original Busemann-Petty problem is affirmative for every L, and, on the other hand, if L is not an intersection body, then one can perturb it to construct a body D giving together with L a counterexample.

Lutwak's result is related to Theorems 1 and 2 via the following Fourier analytic characterization of intersection bodies found by Koldobsky [K3]: an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$.

We present the proof of Theorems 1 and 2 in Section 3. The proof is based on the Fourier transform of distributions, the Spherical Parseval's identity introduced by Koldobsky (see Lemma 3 in [K4] or Proposition 1 in Section 3) and an elementary functional inequality (see Lemma 1).

Another application of Theorem 1 is motivated by a question of what one has to know about the measure of central sections of the bodies K and L to make a conclusion about the relation between the volumes of K and L in every dimension. Results of such a type, involving derivatives or the Laplace transform of the parallel sections functions were proved in [K4], [K6], [K7], [K8], [K9], [RZ], [KYY].

Note that Theorem 1 allows us to start a different approach to this problem which is based on introducing of two different measures: μ_n on convex bodies and μ_{n-1} on hyperplane sections of convex bodies in $\mathbb{R}^n$. This leads to a number of interesting facts and gives examples of non-trivial densities $f_n(x)$ and $f_{n-1}(x)$ for which the BPGM problem has an affirmative answer in any dimension (see Section 4). Probably the most notable statement is (see Corollary 4):

For any $n \geq 2$ and any symmetric star-shaped bodies $K, L \subset \mathbb{R}^n$ such that

$$\int_{K\cap\xi^\perp} \sum_{i=1}^{n} |x_i|\, dx \leq \int_{L\cap\xi^\perp} \sum_{i=1}^{n} |x_i|\, dx$$

for every $\xi \in S^{n-1}$, we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

One of the advantages of the latter result (and other results of this type) is that now one can apply methods and inequalities from asymptotic convex geometry (see [MP]) to produce new bounds on the volume of hyperplane sections of a convex body (see Section 4).

Finally iterating Theorem 1 with different densities f_n and f_{n-1} we present some new results for sections of codimension greater than 1 (see Theorem 4 and Corollary 3).

2. Measure of sections of star-shaped bodies and the Fourier transform

Our main tool is the Fourier transform of distributions (see [GS], [GV] and [K8] for exact definitions and properties). We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$, for every test function ϕ. A distribution f is called *even homogeneous* of degree $p \in \mathbb{R}$ if

$$\langle f(x), \phi(x/t) \rangle = |t|^{n+p} \langle f(x), \phi(x) \rangle, \ \ \forall \phi \in \mathcal{S}, \ t \in \mathbb{R} \setminus \{0\}.$$

The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.

A distribution f is called *positive definite* if, for every nonnegative test function $\phi \in S$,

$$\langle f, \phi * \overline{\phi(-x)} \rangle \geq 0.$$

By Schwartz's generalization of Bochner's theorem, a distribution is positive definite if and only if its Fourier transform is a positive distribution (in the sense that $\langle \hat{f}, \phi \rangle \geq 0$, for every non-negative $\phi \in S$). Every positive distribution is a tempered measure, i.e. a Borel non-negative, locally finite measure γ on $\mathbb{R}^n$ such that, for some $\beta > 0$,

$$\int_{\mathbb{R}^n} (1 + |x|)^{-\beta} d\gamma(x) < \infty,$$

where $|x|$ stands for the Euclidean norm (see [GV] p. 147).

The *spherical Radon transform* is a bounded linear operator on $C(S^{n-1})$ defined by

$$\mathcal{R}f(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x)dx, \;\; f \in C(S^{n-1}), \;\; \xi \in S^{n-1}.$$

Koldobsky ([K1], Lemma 4) proved that if $g(x)$ is an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$ so that $g\big|_{S^{n-1}} \in L_1(S^{n-1})$, then

$$\mathcal{R}g(\xi) = \frac{1}{\pi} \hat{g}(\xi), \qquad \forall \xi \in S^{n-1}. \tag{4}$$

Let K be a body (compact set, with non-empty interior) that is star-shaped with respect to the origin in $\mathbb{R}^n$. The Minkowski functional of K is given by

$$\|x\|_K = \min\{\alpha > 0 : x \in \alpha K\}, \;\; x \in \mathbb{R}^n.$$

Theorem 3. *Let K be a symmetric star-shaped body in $\mathbb{R}^n$, then*

$$\mu_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi} \left(|x|^{-n+1} \int_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt \right)^\wedge (\xi).$$

Proof : If χ is the indicator function of the interval $[-1, 1]$ then, passing to the polar coordinates in the hyperplane $\xi^\perp$ we get

$$\mu_{n-1}(K \cap \xi^\perp) = \int_{(x,\xi)=0} \chi(\|x\|_K) f_{n-1}(x) dx = \int_{S^{n-1} \cap \xi^\perp} \int_0^{\|\theta\|_K^{-1}} t^{n-2} f_{n-1}(t\theta) dt d\theta.$$

We extend the function under the spherical integral to a homogeneous of degree $-n+1$ function on $\mathbb{R}^n$ and apply (4) to get

$$\begin{aligned}
\mu_{n-1}(K\cap\xi^\perp) &= \int\limits_{S^{n-1}\cap\xi^\perp} |x|^{-n+1} \int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dtdx \\
&= \mathcal{R}\left(|x|^{-n+1}\int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)(\xi) \\
&= \frac{1}{\pi}\left(|x|^{-n+1}\int\limits_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi). \quad (5)
\end{aligned}$$

$\square$

Theorem 3 implies that a symmetric star-shaped body is uniquely determined by the measure μ_{n-1} of its sections:

Corollary 1. *Assume $f_{n-1}(x)\neq 0$ everywhere except for a countable set of points in $\mathbb{R}^n$. Let K and L be star-shaped origin symmetric bodies in $\mathbb{R}^n$. If*

$$\mu_{n-1}(K\cap\xi^\perp)=\mu_{n-1}(L\cap\xi^\perp), \quad \forall\xi\in S^{n-1},$$

then $K=L$.

Proof : Note that the function in (5) is homogeneous of degree -1 (with respect to $\xi\in\mathbb{R}^n$). This gives a natural extension of $\mu_{n-1}(K\cap\xi^\perp)$ to a homogeneous function of degree -1. So from the equality of functions $\mu_{n-1}(K\cap\xi^\perp)=\mu_{n-1}(L\cap\xi^\perp)$ on S^{n-1} we get the equality of those functions on $\mathbb{R}^n$:

$$\left(|x|^{-n+1}\int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi) = \left(|x|^{-n+1}\int\limits_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi).$$

Applying the inverse Fourier transform to both sides of the latter equation we get:

$$\int\limits_0^{|x|/||x||_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt = \int\limits_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt, \quad \forall x\in\mathbb{R}^n,$$

which, together with monotonicity of the function $\int\limits_0^y t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt$, for $y\in\mathbb{R}^+$ (because $f_{n-1}(x)>0$ everywhere except for a countable set of points $x\in\mathbb{R}^n$), gives $||x||_K=||x||_L$.

$\square$

3. Proofs of Theorems 1 and 2

We would like to start with the following elementary inequality:

Lemma 1. (Elementary inequality)

$$\int_0^a t^{n-1}\alpha(t)dt - a\frac{\alpha(a)}{\beta(a)}\int_0^a t^{n-2}\beta(t)dt$$
$$\leq \int_0^b t^{n-1}\alpha(t)dt - a\frac{\alpha(a)}{\beta(a)}\int_0^b t^{n-2}\beta(t)dt. \qquad (6)$$

for all $a, b > 0$ and $\alpha(t), \beta(t)$ being nonnegative functions on $(0, \max\{a, b\}]$, such that all integrals in (6) are defined and $t\frac{\alpha(t)}{\beta(t)}$ is increasing on $(0, \max\{a, b\}]$.

Proof : The inequality (6) is equivalent to

$$a\frac{\alpha(a)}{\beta(a)}\int_a^b t^{n-2}\beta(t)dt \leq \int_a^b t^{n-1}\alpha(t)dt.$$

But

$$a\frac{\alpha(a)}{\beta(a)}\int_a^b t^{n-2}\beta(t)dt = \int_a^b t^{n-1}\alpha(t)\left(a\frac{\alpha(a)}{\beta(a)}\right)\left(t\frac{\alpha(t)}{\beta(t)}\right)^{-1}dt \leq \int_a^b t^{n-1}\alpha(t)dt.$$

Note that the latter inequality does not require $a \leq b$.

$\square$

Before proving Theorem 1 (the affirmative case of BPGM) we need to state a version of Parseval's identity on the sphere and to remind a few facts concerning positive definite homogeneous distributions.

Suppose that $f(x)$ is a continuous on $\mathbb{R}^n \setminus \{0\}$ function, which is a positive definite homogeneous distribution of degree -1. Then the Fourier transform of $f(x)$ is a tempered measure γ on $\mathbb{R}^n$ (see Section 2) which is a homogeneous distribution of degree $-n+1$. Writing this measure in the spherical coordinates (see [K5], Lemma 1) we can find a measure γ_0 on S^{n-1} so that for every even test function ϕ

$$\langle \hat{f}, \phi\rangle = \langle \gamma, \phi\rangle = \int_{S^{n-1}} d\gamma_0(\theta)\int_0^\infty \phi(r\theta)dr.$$

Proposition 1. (Koldobsky, [K4]**)** *Let f and g be two functions on $\mathbb{R}^n$, continuous on S^{n-1} and homogeneous of degrees -1 and $-n+1$, respectively. Suppose that f represents a positive definite distribution and γ_0 is the measure on S^{n-1} defined above. Then*

$$\int_{S^{n-1}} \hat{g}(\theta)d\gamma_0(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\theta.$$

Remark: It is crucial that the sum of degrees of homogeneity of the functions f and g is equal to $-n$. This is one of the reasons for the choice of degrees of homogeneity in the conditions (2) and (3) from Theorems 1, and 2 and in the formula from Theorem 3.

Proof of Theorem 1: Consider symmetric star-shaped bodies K and L in $\mathbb{R}^n$, such that

$$\mu_{n-1}(K \cap \xi^{\perp}) \leq \mu_{n-1}(L \cap \xi^{\perp}), \ \ \forall \xi \in S^{n-1}. \tag{7}$$

We apply Theorem 3 to get an analytic form of (7):

$$\left(|x|^{-n+1} \int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt \right)^{\wedge} (\xi) \leq \left(|x|^{-n+1} \int\limits_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt \right)^{\wedge} (\xi).$$

Next we integrate the latter inequality over S^{n-1} with respect to the measure γ_0 corresponding to a positive definite homogeneous of degree -1 distribution (2):

$$\begin{aligned} &\int\limits_{S^{n-1}} \left(|x|^{-n+1} \int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt \right)^{\wedge} (\xi)\, d\gamma_0(\xi) \\ &\qquad \leq \int\limits_{S^{n-1}} \left(|x|^{-n+1} \int\limits_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt \right)^{\wedge} (\xi)\, d\gamma_0(\xi). \end{aligned} \tag{8}$$

Applying the spherical Parseval identity (Proposition 1) we get:

$$\begin{aligned} &\int\limits_{S^{n-1}} \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int\limits_0^{\|x\|_K^{-1}} t^{n-2} f_{n-1}(tx) dt dx \\ &\qquad \leq \int\limits_{S^{n-1}} \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int\limits_0^{\|x\|_L^{-1}} t^{n-2} f_{n-1}(tx) dt dx. \end{aligned} \tag{9}$$

Now we apply Lemma 1, with $a = \|x\|_K^{-1}$, $b = \|x\|_L^{-1}$, $\alpha(t) = f_n(tx)$ and $\beta(t) = f_{n-1}(tx)$ (note that from condition (1) it follows that $t\alpha(t)/\beta(t)$ is

increasing) to get

$$\int_0^{\|x\|_K^{-1}} t^{n-1} f_n(tx)dt - \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int_0^{\|x\|_K^{-1}} t^{n-2} f_{n-1}(tx)dt$$
$$\leq \int_0^{\|x\|_L^{-1}} t^{n-1} f_n(tx)dt - \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int_0^{\|x\|_L^{-1}} t^{n-2} f_{n-1}(tx)dt,\ \forall x \in S^{n-1}.$$

Integrating over S^{n-1} we get

$$\int_{S^{n-1}} \int_0^{\|x\|_K^{-1}} t^{n-1} f_n(tx)dtdx - \int_{S^{n-1}} \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int_0^{\|x\|_K^{-1}} t^{n-2} f_{n-1}(tx)dtdx \tag{10}$$
$$\leq \int_{S^{n-1}} \int_0^{\|x\|_L^{-1}} t^{n-1} f_n(tx)dtdx - \int_{S^{n-1}} \|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})} \int_0^{\|x\|_L^{-1}} t^{n-2} f_{n-1}(tx)dtdx.$$

Adding equations (9) and (10) we get

$$\int_{S^{n-1}} \int_0^{\|x\|_K^{-1}} t^{n-1} f_n(tx)dtdx \leq \int_{S^{n-1}} \int_0^{\|x\|_L^{-1}} t^{n-1} f_n(tx)dtdx,$$

which is exactly $\mu_n(K) \leq \mu_n(L)$.

□

The next proposition is to show that convexity is preserved under small perturbation, which is needed in the proof of Theorem 2.

Proposition 2. *Consider an infinitely smooth origin symmetric convex body L with positive curvature and even functions $f_{n-1}, g \in C^2(R^n \setminus \{0\})$, such that f_{n-1} is strictly positive on $\mathbb{R}^n \setminus \{0\}$. For $\varepsilon > 0$ define a star-shaped body D by an equation for its radial function $\|x\|_D^{-1}$:*

$$\int_0^{\|x\|_D^{-1}} t^{n-2} f_{n-1}(tx)dt = \int_0^{\|x\|_L^{-1}} t^{n-2} f_{n-1}(tx)dt - \varepsilon g(x),\ \ \forall x \in S^{n-1}.$$

Then if ε is small enough the body D is convex.

Proof : For small enough ε, define a function $\alpha_\varepsilon(x)$ on S^{n-1} such that

$$\int_0^{||x||_L^{-1}} t^{n-2} f_{n-1}(tx)dt - \varepsilon g(x) = \int_0^{||x||_L^{-1} - \alpha_\varepsilon(x)} t^{n-2} f_{n-1}(tx)dt,\ \forall x \in S^{n-1}. \tag{11}$$

Using monotonicity of $\int_0^y t^{n-2} f_{n-1}(tx)dt$, for $y \in \mathbb{R}^+$ ($f_{n-1}(tx) > 0$, for $tx \in \mathbb{R}^n \setminus \{0\}$), we get

$$||x||_D^{-1} = ||x||_L^{-1} - \alpha_\varepsilon(x), \ \ \forall x \in S^{n-1}. \tag{12}$$

Moreover, using that $f_{n-1}(x)$, $x \in S^{n-1}$, and its partial derivatives of order one are bounded for finite values of x we get, from (11), that $\alpha_\varepsilon(x)$ and its first and second derivatives converge uniformly to 0 (for $x \in S^{n-1}$, as $\varepsilon \to 0$). Using that L is convex with positive curvature, one can choose a small enough ε so that the body D is convex (with positive curvature).

□

Next we would like to remind a functional version (with an additional differentiability assumption) of Proposition 1.

Proposition 3. (Koldobsky, [K4]**)** *Let f and g be two homogeneous $C^\infty(\mathbb{R}^n\setminus\{0\})$ functions of degrees -1 and $-n+1$, respectively, then*

$$\int_{S^{n-1}} \hat{f}(\theta)\hat{g}(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\theta.$$

We will also need the following fact, which follows from Theorem 1 in [GKS]: if f is positive, symmetric, homogeneous function of degree -1, such that $f(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, then $\hat{f}(x)$ is also an infinitely smooth function on the sphere S^{n-1}.

Proof of Theorem 2: First we will use that function (3) is in $C^\infty(\mathbb{R}^n\setminus\{0\})$, to claim that

$$\left(\|x\|_L^{-1} \frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} \right)^\wedge$$

is a continuous function on S^{n-1}.

This function does not represent a positive definite distribution so it must be negative on some open symmetric subset Ω of S^{n-1}. Consider non-negative even function supported $h \in C^\infty(S^{n-1})$ in Ω. Extend h to a homogeneous function $h(\theta)r^{-1}$ of degree -1. Then the Fourier transform of h is a homogeneous function of degree $-n+1$: $\widehat{h(\theta)r^{-1}} = g(\theta)r^{-n+1}$.

For $\varepsilon > 0$, we define a body D by

$$|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_D}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt$$
$$= |x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt - \varepsilon g\left(\frac{x}{|x|}\right)|x|^{-n+1}.$$

By Proposition 2 one can choose a small enough ε so that the body D is convex.

Since $h \geq 0$, we have

$$\mu_{n-1}(D \cap \xi^{\perp}) = \frac{1}{\pi}\left(|x|^{-n+1}\int_0^{|x|/||x||_D} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi)$$

$$= \frac{1}{\pi}\left(|x|^{-n+1}\int_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi) - \frac{(2\pi)^n \varepsilon h(\xi)}{\pi}$$

$$\leq \mu_{n-1}(L \cap \xi^{\perp}).$$

On the other hand, the function h is positive only where

$$\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)$$

is negative so

$$\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/||x||_D} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi)$$

$$= \left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi)$$

$$-(2\pi)^n\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\varepsilon h(\xi)$$

$$\geq \left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\right)^{\wedge}(\xi).$$

Integrate the latter inequality over S^{n-1} and apply the spherical Parseval identity, Proposition 3. Finally, the same computations (based on Lemma 1) as in the proof of Theorem 1 give

$$\mu_n(D) > \mu_n(L).$$

$\square$

4. Applications

Corollary 2. *Assume $f_n(x) = f_{n-1}(x)$, then the answer to the BPGM problem is affirmative if $n \le 4$ and negative if $n \ge 5$.*

Proof : In this case $tf_n(tx)/f_{n-1}(tx) = t$ is an increasing function, so we may apply Theorems 1 and 2. First note that

$$\|x\|_K^{-1} \frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} = \|\cdot\|_K^{-1}.$$

Thus we may use the fact that for any origin symmetric convex body K in $\mathbb{R}^n$, $n \le 4$, $\|\cdot\|_K^{-1}$ represents a positive definite distribution (see [GKS], [K8]) to give the affirmative answer to BPGM in this case.

For $n \ge 5$, we first note that there is an infinitely smooth, symmetric, convex body $L \subset \mathbb{R}^n$ with positive curvature and such that $\|\cdot\|_L^{-1}$ is not positive definite (see [GKS], [K8]) and thus

$$\|x\|_L^{-1} \frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} = \|\cdot\|_L^{-1}$$

is in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and does not represent a positive definite distribution.

Finally if $f_{n-1} \notin C^\infty(\mathbb{R}^n \setminus \{0\})$ we finish the proof by approximating f_{n-1} (and thus f_n) by a sequence of strictly positive functions belonging to $C^\infty(\mathbb{R}^n \setminus \{0\})$.

□

Remark: Note that the answers for the original Busemann-Petty problem and the Busemann-Petty problem for Gaussian Measures are particular cases of Corollary 2, with $f_n(x) = 1$ and $f_n(x) = e^{-|x|^2/2}$ respectively.

Lemma 2. *Consider a symmetric star-shaped body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite, then for any symmetric star-shaped bodies $K, L \subset \mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$\int_{K\cap\xi^\perp} \|x\|_M dx \le \int_{L\cap\xi^\perp} \|x\|_M dx \tag{13}$$

we have

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L).$$

Proof : This follows from Theorem 1 with $f_n = 1$, $f_{n-1} = \|x\|_M$. In this case,

$$\|x\|_K^{-1} \frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} = \|x\|_K^{-1} \frac{1}{\|\frac{x}{\|x\|_K}\|_M} = \|\cdot\|_M^{-1}.$$

□

Lemma 3. *Consider a symmetric star-shaped body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite. Then for any symmetric star-shaped bodies $K, L \subset \mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$\mathrm{Vol}_{n-1}(K \cap \xi^{\perp}) \leq \mathrm{Vol}_{n-1}(L \cap \xi^{\perp}), \tag{14}$$

we have

$$\int_K \|x\|_M^{-1} dx \leq \int_L \|x\|_M^{-1} dx$$

Proof : This theorem follows by the same argument as in Lemma 2, but with the functions $f_n = \|x\|_M^{-1}$, $f_{n-1} = 1$.

□

Remark: It follows from Theorem 2, and the standard approximation argument, that Lemmas 2 and 3 are not true (even, with additional convexity assumption) if $\|x\|_M^{-1}$ is not positive definite.

Next we will use Theorem 1 and ideas from Lemmas 2 and 3 to show some results on a lower dimensional version of the BPGM.

Theorem 4. *Consider a symmetric star-shaped body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite, then for any symmetric star-shaped bodies $K, L \subset \mathbb{R}^n$, and $1 \leq k < n$ such that, for every $H \in G(n, n-k)$*

$$\int_{K\cap H} ||x||_M^k dx \leq \int_{K\cap H} ||x||_M^k dx,$$

we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : It was proved in [GW] that every hyperplane section of intersection body is also an intersection body. Using the relation of positive definite distributions to intersection bodies we get that if $\|x\|_M^{-1}$ is a positive definite distribution than the restriction of $\|x\|_M^{-1}$ on subspace F, is also a positive definite distribution. Thus we may apply Theorem 1 with functions $f_{i-1}(x) = ||x||_M^{n-i+1}$ and $f_i(x) = ||x||_M^{n-i}$. Indeed, in this case

$$\|x\|_K^{-1} \frac{f_i\left(\frac{x}{\|x\|_K}\right)}{f_{i-1}\left(\frac{x}{\|x\|_K}\right)} = \|x\|_K^{-1} \frac{\|\frac{x}{\|x\|_K}\|_M^{n-i}}{\|\frac{x}{\|x\|_K}\|_M^{n-i+1}} = ||\cdot||_M^{-1}$$

is a positive definite distribution. So, from Theorem 1, we get that if for every $H \in G(n, n-i)$

$$\int_{K\cap H} ||x||_M^i dx \leq \int_{K\cap H} ||x||_M^i dx,$$

then for every $F \in G(n, n-i+1)$

$$\int_{K\cap F} ||x||_M^{i-1} dx \leq \int_{K\cap F} ||x||_M^{i-1} dx.$$

We iterate this procedure for $i = k, k-1, \ldots, 1$ to finish the proof.

□

We can present a different version of Theorem 4, in a special cases of $n-k = \{2,3\}$ and K is a convex symmetric body:

Corollary 3. *Consider a symmetric star-shaped body $M \subset \mathbb{R}^n$, $n \geq 4$, such that $\|x\|_M^{-1}$ is positive definite. Fix k such that $n-k \in \{2,3\}$, then for any convex symmetric bodies $K, L \subset \mathbb{R}^n$, such that, for every $H \in G(n, n-k)$*

$$\int_{K\cap H} ||x||_M^{n-4} dx \leq \int_{K\cap H} ||x||_M^{n-4} dx, \tag{15}$$

we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : We use the same iteration procedure as in Theorem 4, but the first steps of iteration procedure, for example, from subspaces of dimension 2 to subspaces of dimension 3 is different and we use Corollary 2 with $f_2(x) = f_3(x) = ||x||_M^{n-4}$. We use the same idea to iterate from dimension 3 to 4.

□

Remark: Note that if $n-k=2$, then Corollary 3 is still true with power $n-3$, instead of $n-4$ in (15). Corollary 3 is a generalization of a result of Koldobsky ([K6], Theorem 8; see also [RZ]), where the case of $n-k=3$ and $M = B_2^n$ was considered.

We also note that the generalization of the standard Busemann-Petty problem is open for those dimensions (see [BZ], [K6], [RZ]).

Let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, $\|x\|_\infty = \max\limits_i |x_i|$ and $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$.

Corollary 4. *Consider $p \in (0,2]$ and $n \geq 2$. Then for any symmetric star-shaped bodies $K, L \subset \mathbb{R}^n$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_p dx \leq \int_{L\cap\xi^\perp} \|x\|_p dx \tag{16}$$

for every $\xi \in S^{n-1}$, we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : This follows from Lemma 2 and the fact that $\|x\|_p^{-1}$ is positive definite for $p \in (0,2]$ (see [K3], [K8]).

□

Remark: Note that $\|x\|_p^{-1}$ does not represent a positive definite distribution when $p \in (2,\infty]$ and $n > 4$ (see [K3], [K8]), thus applying Theorem 2, together with the standard approximation argument, we get that the statement of Corollary 4 is not true in those cases, and counterexamples (even with bodies K and L being convex) can be constructed.

Next we would like to use Lemma 2 to give a lower bound on the integral over a hyperplane section of the convex body.

Lemma 4. *Consider a symmetric star-shaped body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite. Then for any star-shaped body $K \subset \mathbb{R}^n$ there exits a direction $\xi \in S^{n-1}$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_M dx \geq \frac{n-1}{n}\frac{\mathrm{Vol}_{n-1}(M\cap\xi^\perp)}{\mathrm{Vol}_n(M)}\mathrm{Vol}_n(K).$$

Proof : Assume it is not true, then

$$\int_{K\cap\xi^\perp} \|x\|_M dx < \frac{n-1}{n}\frac{\mathrm{Vol}_{n-1}(M\cap\xi^\perp)}{\mathrm{Vol}_n(M)}\mathrm{Vol}_n(K), \ \ \forall \xi \in S^{n-1}.$$

Also note that for $L \subset \mathbb{R}^{n-1}$

$$\int_L \|x\|_L dx = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta = \frac{n-1}{n}\mathrm{Vol}_{n-1}(L).$$

Applying the latter equality to $L = M \cap \xi^\perp$ to get

$$\int_{K\cap\xi^\perp} \|x\|_M dx < \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)}\int_{M\cap\xi^\perp} \|x\|_M dx, \ \ \forall \xi \in S^{n-1}.$$

Let $r^n = \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)}$, then

$$\int_{K\cap\xi^\perp} \|x\|_{rM} dx < \int_{rM\cap\xi^\perp} \|x\|_{rM} dx, \ \ \forall \xi \in S^{n-1}.$$

Thus

$$\mathrm{Vol}_n(K) < \mathrm{Vol}(rM),$$

or

$$\mathrm{Vol}_n(K) < \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)}\mathrm{Vol}_n(M)$$

which gives a contradiction.

□

Corollary 5. *For any $p \in [1,2]$ and any symmetric star-shaped body K in $\mathbb{R}^n$ there exists a direction $\xi \in S^{n-1}$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_p dx \geq c_p n^{1/p}\mathrm{Vol}_n(K),$$

where c_p is a constant depending on p only.

Proof : Again $\|x\|_p^{-1}$, $p \in [1,2]$ is a positive definite distribution (see [K3], [K8]) and thus we may apply Lemma 4 (or Corollary 4) to get that there exist $\xi \in S^{n-1}$ such that

$$\int_{K\cap\xi^\perp} \|x\|_p dx \geq \frac{n-1}{n}\frac{\mathrm{Vol}_{n-1}(B_p^n\cap\xi^\perp)}{\mathrm{Vol}_n(B_p^n)}\mathrm{Vol}_n(K).$$

Next we use that B_p^n, $p \in [1,2]$ is in the isotropic position, and the isotropic constant $L_{B_p^n} \leq c$ (see [Sc]), thus the ratio of volume of different hyperplane

sections is bounded by two universal constants (see [Bo1]; [MP], Corollary 3.2):

$$c \le \frac{\mathrm{Vol}_{n-1}(B_p^n \cap \xi^\perp)}{\mathrm{Vol}_{n-1}(B_p^n \cap \nu^\perp)} \le C, \ \ \forall \xi, \nu \in S^{n-1},$$

and

$$c\mathrm{Vol}_{n-1}(B_p^{n-1}) \le \mathrm{Vol}_{n-1}(B_p^n \cap \xi^\perp) \le C\mathrm{Vol}_{n-1}(B_p^{n-1}), \ \ \forall \xi \in S^{n-1}.$$

Applying

$$\mathrm{Vol}_n(B_p^n) = \frac{[2\Gamma(1+\frac{1}{p})]^n}{\Gamma(1+\frac{n}{p})},$$

we get

$$\frac{\mathrm{Vol}_{n-1}(B_p^{n-1})}{\mathrm{Vol}_n(B_p^n)} = \frac{1}{2\Gamma(1+\frac{1}{p})}\frac{\Gamma(1+\frac{n}{p})}{\Gamma(1+\frac{n-1}{p})} \ge c_p n^{1/p}.$$

□

Corollary 6. *For any convex symmetric body $K \in \mathbb{R}^n$, there are vectors $\xi, \nu \in S^{n-1}$ such that $\xi \ne \pm\nu$ and*

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) \ge c\sqrt{\mathrm{Vol}_{n-2}(K \cap \{\xi,\nu\}^\perp)\mathrm{Vol}_n(K)},$$

where $\{\xi,\nu\}^\perp$ is the subspace of codimension 2, orthogonal to ξ and ν.

Proof : From Corollary 5 (with $p = 1$) we get that there exists a direction $\xi \in S^{n-1}$ such that

$$\int_{K\cap\xi^\perp} \sum_{i=1}^n |x_i| dx \ge cn\mathrm{Vol}_n(K). \tag{17}$$

From continuity of volume measure, we may assume that $\xi \notin \{\pm e_i\}_{i=1}^n$, where $\{e_i\}_{i=1}^n$ is the standard basis in $\mathbb{R}^n$. From the inequality (17) we get that there exists $j \in \{1, \dots, n\}$ such that

$$\int_{K\cap\xi^\perp} |x_j| dx \ge c\mathrm{Vol}_n(K). \tag{18}$$

Next we apply the "inverse Holder" inequality ([MP], Corollary 2.7; see also [GrM]): for any symmetric convex body L in $\mathbb{R}^{n-1}$ and unit vector $\theta \in S^{n-2}$

$$\int_L |x \cdot \theta| dx \le c\frac{\mathrm{Vol}_{n-1}^2(L)}{\mathrm{Vol}_{n-2}(L \cap \theta^\perp)},$$

Using the latter inequality for $L = K \cap \xi^\perp$ and $\theta = e_j$ we get

$$\int_{K\cap\xi^\perp} |x_j| dx \le c\frac{\mathrm{Vol}_{n-1}^2(K \cap \xi^\perp)}{\mathrm{Vol}_{n-1}(K \cap \{\xi, e_j\}^\perp)}. \tag{19}$$

We compare inequalities (18) and (19) to finish the proof.

□

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